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**An axiomatic approach to utilitarianism**

**Abstract**

We present first a simple axiomatization of utilitarianism and then an axiomatisation of generalized utilitarianism that does not rely on any independence-by-transformation assumption.

# 1 Introduction

$N = \{1, \ldots, n\}$ is the set of individuals. $X$ is a set of social alternatives. We assume that each individual $i \in N$ has a utility function $U_i : X \to \mathbb{R}$. We denote by $U = (U_1, \ldots, U_n)$ a utility profile. We consider $\mathcal{D} = (\mathbb{R}^X)^n$ : the set of all vectors of utility functions.
We are interested in $F : \mathcal{D} \to \mathcal{O}(X)$, $F$ is the social welfare function, which associates to each utility profile an ordering (here : reflexive, transitive and complete binary relation) on the set of alternatives $X$ .We denote by $R_U = F(U)$ the social ordering on $X$ associated to the utility profile $U$.
Our first assumption is the following : there exists a social-evaluation ordering $R$ on $\mathbb{R}^n$ such that :

$$\forall x, y \in X,\ \forall U \in \mathcal{D}, \qquad x R_U y \Leftrightarrow U(x) R U(y)$$

This assumption is called *welfarism*: to choose between $x$ and $y$, the social planner takes only into account the utilities of the individuals in alternatives $x$ and $y$. We associate to $R$ the strict relation $P$ and the indifference relation $I$.

# 2 A simple characterization of utilitarianism

## 2.1 Axioms

We assume introduce some axioms on $R$:

**Axiom 1.** *Anonymity(A)*

$$\forall \pi \in \Sigma_n, \forall u, v \in \mathbb{R}^n, \qquad uRv \Leftrightarrow \pi(u) R \pi(v)$$

*with* $\pi(w) = (w_{\pi(1)}, \ldots, w_{\pi(n)})$.

**Axiom 2.** *Information invariance with respect to translation-scale measurable utilities (TSM)*

$$\forall u, v, a \in \mathbb{R}^n, \qquad uRv \Leftrightarrow (u+a)R(v+a).$$

**Axiom 3.** *Weak Pareto (WP)*
*For all $u, v \in \mathbb{R}^n$, if $u_i > v_i$ for all $i \in N$, then $uPv$.*

## 2.2 The result

**Theorem 1.** *(Utilitarianism)*
*The social-evaluation ordering $R$ satisfies the axioms (A),(TSM) and (WP) if and only if $R$ can be represented by the utilitarian criterion:*

$$uRv \Leftrightarrow \sum_{i=1}^{n} u_i \geq \sum_{i=1}^{n} v_i.$$

## 2.3 Proof

We know from axioms (A) and (TSM) that $R$ satisfies the following axiom :

**Axiom 4.** *Incremental Equity (IE)*

$$\forall u \in \mathbb{R}^n, \forall \delta \in \mathbb{R}, \forall j, k \in N, \qquad (u + \delta \mathbb{1}_n^j) I (u + \delta \mathbb{1}_n^k)$$

Then we apply (IE) to $u - \delta \mathbb{1}_n^j$. We get

$$\forall j, k \in N, \forall u \in \mathbb{R}^n, \delta \in \mathbb{R}, \qquad uI(u - \delta \mathbb{1}_n^j + \delta \mathbb{1}_n^k)$$

We obtain for a given $u \in \mathbb{R}^n$:

$$\begin{aligned}
&uI\left(\frac{1}{n}\sum_{i=1}^{n} u_i, u_2, \ldots, u_n + u_1 - \frac{1}{n}\sum_{i=1}^{n} u_i\right) \\
&I\left(\frac{1}{n}\sum_{i=1}^{n} u_i, \frac{1}{n}\sum_{i=1}^{n} u_i, u_3, \ldots, u_n + u_1 + u_2 - \frac{2}{n}\sum_{i=1}^{n} u_i\right) \\
&I \ldots \\
&I\left(\frac{1}{n}\sum_{i=1}^{n} u_i, \ldots, \sum_{i=1}^{n} - \frac{n-1}{n}\sum_{i=1}^{n} u_i\right) \\
&I\left(\frac{1}{n}\sum_{i=1}^{n} u_i\right) \mathbb{1}_n
\end{aligned}$$

The conclusion comes from the weak axiom of Pareto (WP).

# 3 The theorem of generalized utilitarianism.

## 3.1 Axioms

**Axiom 5.** *Anonymity(A)*

$$\forall \pi \in \Sigma_n, \qquad uRv \Leftrightarrow \pi(u)R\pi(v)$$

*with* $\pi(w) = (w_{\pi(1)}, \ldots, w_{\pi(n)})$.

**Axiom 6.** *Continuity (C)*
*For all* $u \in \mathbb{R}^n$, $\{v|uRv\}$ *and* $\{v|vRu\}$ *are closed subsets of* $\mathbb{R}^n$.

**Axiom 7.** *Strong Pareto (SP)*
*For all* $u, v \in \mathbb{R}^n$, *if* $u_i \geq v_i$ *for all* $i \in N$ *with at least one strict inequality, then* $uPv$.

The crucial axiom that we use to get the additivity result is :

**Axiom 8.** *Independence of the Utilities of the Unconcerned Individuals (IUUI)*
*For all* $M \subset N$, *for all* $u, v, u', v' \in \mathbb{R}^n$, *if*

$$\begin{array}{rlr} \forall i \in M, & u_i = v_i, & u'_i = v'_i \\ \forall j \in N \backslash M, & u_j = u'_j, & v_j = v'_j \end{array}$$

*then* $uRv \Leftrightarrow u'Rv'$.

In this definition, $M$ is the set of unconcerned individuals.

## 3.2 The result

**Theorem 2.** *(Generalized Utilitarianism)*
*The social-evaluation ordering* $R$ *satisfies the axioms (A),(C),(SP) and (IUUI) if and only if there exists a continuous and increasing function* $g : \mathbb{R} \to \mathbb{R}$ *such that :*

$$uRv \Leftrightarrow \sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{n} g(v_i).$$

# 4 Proof

We assume now that $R$ verifies the axioms (A),(C),(SP),(IUUI).

### 4.1 Part 1 : Debreu's representation theorem

**Definition 1.** *We say that a binary relation $R$ on a set $S$ is represented by a function $W : S \to \mathbb{R}$ if:*

$$\forall u, v \in S, \qquad uRv \Leftrightarrow W(u) \geq W(v).$$

**Theorem 3.** *A complete and transitive relation $R$, defined on $\mathbb{R}^n$, can be represented by a continuous utility function $W : \mathbb{R}^n \to \mathbb{R}$ if and only if $R$ satisfies the axiom (C).*

This result is obtained by Debreu in his "Theory of value"([3]).
Let's denote by $S = \mathbb{R}^n$ the set on which $R$ is defined. We know that $S$ is separable : this means that there exists a dense and countable subset $D \subset S$.
We will note in this proof $v \prec u$ for $uPv$, $v \preceq u$ for $uRv$ and $v \sim u$ for $uIv$.

(1) **A first remark :**

$$\forall s', s'' \in S, s' \prec s'', \qquad \exists s \in D, s' \prec s \prec s''$$

We know that $S_{s'} = \{s \in S | s \prec s'\}$ and $S^{s''} = \{s \in S | s'' \prec s\}$ are disjoint (by transitivity), non-empty and closed in $S$ (by continuity of $R$). Since $S$ is connected, we have that $S_{s'} \cup S^{s''} \neq S$. If $D$ was included in $S_{s'} \cup S^{s''}$, his closure $S$ would be included in $\bar{S_{s'}} \cup \bar{S^{s''}} = S_{s'} \cup S^{s''}$. Hence, $D$ is not included in this set.

(2) **Definition of a function $W'$ on $D$.**
If $D$ has a least element $d^L$, we set $W'(d^L) = 0$. If $D$ has a greatest element $d^G$, we set $W'(d^G) = 1$. $D'$ is the set obtained after removing all elements of $D$ that are indifferent to $d^L$ or $d^G$.
We know by (1) that $D'$ has no greatest or least element. We note : $D' = \{d_k\}_{k \in \mathbb{N}}$ and $\mathbb{Q} \cap ]0, 1[ = \{r_k\}_{k \in \mathbb{N}}$. We define

$$W' : \left| \begin{array}{ll} D' & \to \mathbb{Q} \cap ]0, 1[ \\ d_k & \mapsto r_{\phi(k)} \end{array} \right.$$

by the following algorithm :

- For $d_1$ : $\phi(1) = 1$, $W'(d_1) = r_1$.
- For $d_p$ : Either there is some $p' < p$ such that $d_p \sim d_{p'}$. In this case we set $\phi(p) = \phi(p')$, $W'(d_p) = W'(d_{p'})$.

  Either there are some $p', p'' < p$ such that $d_{p'} \prec d_p \prec d_{p''}$. In this case we choose the rational of least rank in the interval $]r_{\phi(p')}, r_{\phi(p'')}[$, this defines $\phi(p)$ (the rank of this rational) and $W'(d_p) = r_{\phi(p)}$.

The function $W'$ verifies the following properties :

- $W'(D') = \mathbb{Q} \cap ]0, 1[$, from (1).

- $W'$ is increasing.

(3) **Definition of $W$ on $S$.**
If $s$ is a least element of $S$, we set $W(s) = 0$. If $s$ is a greatest element of $S$, we set $W(s) = 1$.
For $s \in S$, we define $D_s = \{d \in D | d \preceq s\}$ and $D^s = \{d \in D | s \preceq d\}$.
By transitivity of $R$ and monotonicity of $W'$, we know that $\sup W'(D_s) \leq \inf W'(D^s)$. Moreover, if these two numbers were different, there would be a rational between them that would not appear in $W'(D)$, which is impossible. Then, we set :

$$W(s) = \sup W'(D_s) = \inf W'(D^s)$$

The function $W$ verifies the following properties :

- $W_{|D} = W'$
- $W$ is increasing.

(4) **Continuity of $W$.**
Let's show that for all $c \in \mathbb{R}$, $W^{-1}([c, \infty[)$ is closed. For $t \in \mathbb{R}$, we note $S_t = \{s \in S | W(s) \leq t\}$ and $S^t = \{s \in S | W(s) \geq t\}$. By density of $\mathbb{Q}$ in $\mathbb{R}$, we have for $c \in ]0, 1[$,

$$S^c = \cap_{r \in \mathbb{Q}, r \leq c} S^r.$$

For $r \in \mathbb{Q} \cap ]0, 1[$, there exists $s \in S$ such that $W(s) = r$. Since $W$ is increasing, we have $S^r = \{s' | s \preceq s'\}$, which is a closed subset by continuity of $R$. This implies that $S^c$ is closed for any real $c$. Then, $W$ is continuous.

## 4.2 Part 2 : the fundamental theorem of Gorman

The results of this section are demonstrated by Gorman ([2]). In the words of Gorman, axiom (IUUI) implies separability of all sectors, and (SP) implies that all the sectors are strictly essential.

**Lemma 1.** *For all partition $N = I \cup J$, $s = (s_I, s_J)$, we can write*

$$W(s) = F(W_I(s_I), W_J(s_J))$$

*with*

- *$F$ continuous and strictly increasing.*
- *$W_I(s_I) = W(s_I, 0)$, $W_J(s_J) = W(0, s_J)$ and $W(0) = 0$.*

(1) **Existence of $F$.**
We set $W(0) = 0$ by translation and define : $W_I(s_I) = W(s_I, 0)$, $W_J(s_J) = W(0, s_J)$. We take $s', s'' \in S$ and consider the vectors :

$$\begin{aligned} s^1 = s' &= (s'_I, s'_J) \\ s^2 &= (s''_I, s'_J) \\ s^3 = s'' &= (s''_I, s''_J) \end{aligned}$$

Assume that $W_I(s'_I) = W_I(s''_I)$ and $W_J(s'_J) = W_J(s''_J)$. Then, by (IUUI) we obtain that $W(s^1) = W(s^2) = W(s^3)$, so that $W(s') = W(s'')$. This means that we can write :

$$\forall s, \qquad W(s) = F(W_I(s_I), W_J(s_J))$$

By the same reasoning, we know that if we have $W_I(s'_I) \geq W_I(s''_I)$ and $W_J(s'_J) \geq W_J(s''_J)$ with at least one strict inequality, we must have $W(s') > W(s'')$ by (IUUI).
Then, $F$ is strictly increasing.

(2) **Continuity of $F$ (technical).**
We note $W_{I,J} = (W_I, W_J)$ a vector in $(\mathbb{R}^{|I|}) \times (\mathbb{R}^{|J|})$. We take a convergent sequence $(W^t_{I,J})_{t\in\mathbb{N}}$, which converges to $W_{I,J}$.
Using axiom (SP), we can choose for all $K \in \{I, J\}$, $s'_K, s''_K \in \mathbb{R}^{|K|}$ such that

$$W_K(s'_K) < W_K < W_K(s''_K)$$

Let $C_K$ be an arc connecting $s'_K$ to $s''_K$. We know that there exists some $T$ such that

$$\forall t \geq T,\ \forall K \in \{I, J\},\ W_K(s'_K) < W^t_K < W_K(s''_K)$$

Because each $W_K$ is continuous, we can find $s^t_K \in C_K$ such that $W^t_K = W_K(s^t_K)$. We now have a sequence $(s^t = (s^t_I, s^t_J))_{t\in\mathbb{N}}$ in the compact set $C_I \times C_J$. Take $s$ an adherence value of $(s^t)$.
We know by construction of $(s^t)$ that $F(W^t_{I,J}) = W(s^t)$. Moreover, by continuity of the $\{W_K\}_{K=I,J}$, we obtain that the subsequence $(W^t_{I,J})$ converges to $W_{I,J}(s)$, so that $W_{I,J}(s) = W_{I,J}$.
Now, by continuity of $W$, we know that the subsequence $(F(W^t_{I,J}))$ converges to $W(s) = F(W_{I,J})$. Thus, the only adherence value of $(F(W^t_{I,J}))$ is $F(W_{I,J})$. We conclude by noting that $F(W^t_{I,J}) \in W(C_I \times C_J)$, which is compact.

**Lemma 2.** *If $A, B \subset N$ overlap, with $A \cup B = N$,*

$$W(s_{A\setminus B}, s_{A\cap B}, s_{B\setminus A}) = F(a(s_{A\setminus B}) + b(s_{A\cap B}) + c(s_{B\setminus A}))$$

*with $F, a, b, c$ continuous and increasing functions.*

For sake of simplicity, we note : $x = s_{A\setminus B}$, $y = s_{A\cap B}$, $z = s_{B\setminus A}$. We know that there exists some continuous and increasing functions $G$ and $H$ such that :

$$\forall s = (x, y, z), \qquad W(x, y, z) = G\big(W_A(x, y), W_{B\setminus A}(z)\big) = H\big(W_{A\setminus B}(x), W_B(y, z)\big)$$

Moreover, we can apply the previous lemma on the set $A$ and $B$, so that there exists continuous and increasing functions $g$ and $h$ such that :

$$\begin{aligned} \forall (x, y), \qquad & W_A(x, y) = g(W_{A\setminus B}(x), W_{A\cap B}(y)) \\ \forall (y, z), \qquad & W_B(y, z) = h(W_{A\cap B}(y), W_{B\setminus A}(z)) \end{aligned}$$

We finally obtain the following functional equation:

$$\forall s = (x, y, z), \qquad W(x, y, z) = G\big(g(u(x), v(y)), w(z)\big) = H\big(u(x), h(v(y), w(z))\big)$$

with $G, H, g, h, u, v, w$ continuous and increasing functions.
The application of a theorem of Aczel[1] implies that :

$$\forall s = (x, y, z), \qquad W(x, y, z) = F\big(a(u(x)) + b(v(y)) + c(w(z))\big)$$

with $F, a, b, c$ continuous and increasing functions.

**Lemma 3.** *The additivity can be extended : $R$ can be represented by a continuous and increasing function $W$ such that :*

$$W(s) = \sum_{i=1}^{n} W_i(s_i)$$

*where* $\forall i \in N,\ W_i(s_i) = W(0, \ldots, 0, s_i, 0, \ldots, 0)$.

(1) **A first remark**
Let $\{I, J\}$; $\{I', I''\}$; $\{J', J''\}$ be partitions of $N$;$I$;$J$. We know from the previous lemma that the social-evaluation ordering $R$ can be represented by :

$$\bar{W}(s) = a\big(W_{I'}(s_{I'})\big) + b\big(W_{I''}(s_{I''})\big) + c\big(W_J(s_J)\big)$$

After a translation of $(a(0) + b(0) + c(0))$, this can be re-written as :

$$\bar{W}(s) = \bar{W}_{I'}(s_{I'}) + \bar{W}_{I''}(s_{I''}) + \bar{W}_J(s_J) = \bar{W}_I(s_I) + \bar{W}_J(s_J)$$

[1] In the statement of Aczel's theorem, the functions have to be defined on the same nondegenerate interval. This is not *a priori* true here, but this can be obtained by renormalization.

But $R$ can also be represented by :

$$\tilde{W}(s) = \tilde{W}_I(s_I) + \tilde{W}_{J'}(s_{J'}) + \tilde{W}_{J''}(s_{J''}) = \tilde{W}_I(s_I) + \tilde{W}_J(s_J)$$

We know that there exists some increasing function $F$ such that $\tilde{W} = F(\bar{W})$. Moreover, the function $F$ is continuous (this is the same proof as for lemma 2).
We can now write : $F(\bar{W}_I + \bar{W}_J) = \tilde{W}_I + \tilde{W}_J = F(\bar{W}_I) + F(\bar{W}_J)$. The solution of this simple functional equation is $F = \lambda Id$, with $\lambda > 0$. This implies finally that $\tilde{W}_I = \tilde{W}_{I'} + \tilde{W}_{I''}$ and we get :

$$\tilde{W} = \tilde{W}_{I'} + \tilde{W}_{I''} + \tilde{W}_{J'} + \tilde{W}_{J''}$$

(2) **Extension by induction**
We want to show that for each $k$, we have $W = W_1 + \ldots + W_k + W_J$, with $J = \{k+1, \ldots, n\}$.

- For $k = 2$, this is already done.
- Assume it is true for some rank $k \geq 2$. Then take $I' = \{1, \ldots, k-1\}$, $I'' = \{k\}$, $J' = \{k+1\}$, then the remark allows to conclude that the property holds at rank $k+1$.

### 4.3 Aczel's theorem

**Theorem 4.** *(Aczel)*
*The general solution of the equation*

$$W(x, y, z) = G\big(g(x, y), z\big) = H\big(x, h(y, z)\big) \qquad (E)$$

*with all functions continuous and increasing, defined on the same non-degenerate interval $I$ is:*

$$\begin{aligned} g(x, y) &= \phi(a(x) + b(y)) \\ h(y, z) &= \psi(b(y) + c(z)) \\ G(g, z) &= F(\phi^{-1}(g) + c(z)) \\ H(x, h) &= F(a(x) + \psi^{-1}(h)) \end{aligned}$$

*with all functions continuous and increasing. This implies that $W = F(a(x) + b(y) + c(z))$.*

Take a number $a \in I$. If we fix $x = z = a$, we obtain :

$$\forall y, \qquad G\big(g(a, y), a\big) = H\big(a, h(y, a)\big).$$

Now define $g_2 = g(a, \cdot), h_1 = h(\cdot, a), G_1 = G(\cdot, a), H_2 = H(a, \cdot)$. The previous equation can be written as :

$$\forall y, \qquad G_1 \circ g_2(y) = H_2 \circ h_1(y) \qquad (*)$$

We will write three equalities, applying (E) three times, to obtain our result.
If we set $x = a$, $y = g_2^{-1} \circ G_1^{-1}(u)$, $z = h_2^{-1} \circ H_2^{-1}(v)$, we get

$$\begin{aligned} G\big(G_1^{-1}(u), h_2^{-1} \circ H_2^{-1}(v)\big) &= G\big(g(a,y), z\big) \\ &= H\big(a, h(y,z)\big) \\ &= H_2 \circ h\big(g_2^{-1} \circ G_1^{-1}(u), h_2^{-1} \circ H_2^{-1}(v)\big) \quad (1) \end{aligned}$$

If we set $x = g_1^{-1} \circ G_1^{-1}(u)$, $y = a$, $z = h_2^{-1} \circ H_2^{-1}(v)$, we get

$$\begin{aligned} G\big(G_1^{-1}(u), h_2^{-1} \circ H_2^{-1}(v)\big) &= G\big(g(x,a), z\big) \\ &= H\big(x, h(a,z)\big) \\ &= H\big(g_1^{-1} \circ G_1^{-1}(u), H_2^{-1}(v)\big) \qquad (2) \end{aligned}$$

If we set $x = g_1^{-1} \circ G_1^{-1}(u)$, $y = h_1^{-1} \circ H_2^{-1}(v)$, $z = a$, we get

$$\begin{aligned} G_1 \circ g\big(g_1^{-1} \circ G_1^{-1}(u), h_1^{-1} \circ H_2^{-1}(v)\big) &= G\big(g(x,y), a\big) \\ &= H\big(x, h(y,a)\big) \\ &= H\big(g_1^{-1} \circ G_1^{-1}(u), H_2^{-1}(v)\big) \quad (3) \end{aligned}$$

We finally obtain :

$$H_2 \circ h\big(g_2^{-1} \circ G_1^{-1}(u), h_2^{-1} \circ H_2^{-1}(v)\big) = G_1 \circ g\big(g_1^{-1} \circ G_1^{-1}(u), h_1^{-1} \circ H_2^{-1}(v)\big)$$

By (*), we can note $\rho(w) = g_2^{-1} \circ G_1^{-1}(w) = h_1^{-1} \circ H_2^{-1}(w)$, so that the previous equation becomes

$$H_2 \circ h\big(\rho(u), h_2^{-1} \circ H_2^{-1}(v)\big) = G_1 \circ g\big(g_1^{-1} \circ G_1^{-1}(u), \rho(v)\big) := uTv \qquad (4)$$

Let's now define : $a = G_1 \circ g_1,\ b = \rho^{-1},\ c = H_2 \circ h_2,\ \phi = G_1^{-1},\ \psi = H_2^{-1}$, we obtain :

$$\begin{aligned} &(4) \Rightarrow g(x,y) = \phi(a(x)Tb(y)) \\ &(4) \Rightarrow h(y,z) = \psi(b(y)Tc(z)) \\ &(1) \Rightarrow G(g,z) = \phi^{-1}(g)Tc(z) \\ &(3) \Rightarrow H(x,h) = a(x)T\psi^{-1}(z) \end{aligned}$$

By re-writting equation (E) with these new notations, we obtain that the binary operation $T$ is associative :

$$G\big(g(x,y),z\big) = \big(a(x)Tb(y)\big)Tc(z)$$
$$H\big(x,h(y,z)\big) = a(x)T\big(b(y)Tc(z)\big)$$

The conclusion is obtained with the following lemma :

**Lemma 4.** *(Aczel)*
*Let $I$ be a non-degenerate interval. Any associative binary operation $T : I^2 \to I$ that is continuous and strictly increasing is such that there exists a continuous and monotonic function $f : J \to I$ such that :*

$$uTv = f[f^{-1}(u) + f^{-1}(v)]$$

*for a non-degenerate interval $J$.*

### 4.4 The associativity equation (Aczel)

We need to show that there exists a continuous and monotonic semigroup homomorphism $f : (J,+) \to (I,T)$ for some interval $J \subset \mathbb{R}$.
We write $I =< a,b >$ with $a,b \in \bar{\mathbb{R}}$.

(1) **Typology of elements of $I$**
An element $c \in I$ can be of three types :

$$cTc > c \qquad (1)$$
$$cTc = c \qquad (2)$$
$$cTc < c \qquad (3)$$

If $c$ is of type (1), because $T$ is increasing, we have for any $x \in I$,

$$cT(cTx) = (cTc)Tx > cTx.$$

Thus, $cTx > x$. We obtained

$$(1) \Rightarrow \forall x \in I,\ cTx > x$$

For similar reasons, it holds that

$$(1) \Leftrightarrow \forall x \in I,\ cTx > x \text{ and } xTc > x \qquad (1')$$
$$(2) \Leftrightarrow \forall x \in I,\ cTx = x \text{ and } xTc = x \qquad (2')$$
$$(3) \Leftrightarrow \forall x \in I,\ cTx < x \text{ and } xTc < x \qquad (3')$$

If there exists some $x_0$ such that $cTx_0 > x_0$, then we have

$$(cTc)Tx_0 = cT(cTx_0) > cTx_0,$$

and then $cTc > c$.
For similar reasons, it holds that

$$\begin{aligned}
(1) &\Leftrightarrow \exists x \in I,\ cTx > x \text{ or } xTc > x && (1'') \\
(2) &\Leftrightarrow \exists x \in I,\ cTx = x \text{ or } xTc = x && (2'') \\
(3) &\Leftrightarrow \exists x \in I,\ cTx < x \text{ or } xTc < x && (3'')
\end{aligned}$$

Notice that there can be at most one element of type (2) in $I$, we call it the identity element of $I$ if it exists. Since $I$ is non-degenerate, there must be elements of type (1) or of type (3). **We assume for the remaining part of the proof that there exists $c \in I$ such that $cTc > c$ (type (1)).**

(2) **Three different cases**
Assume that $I$ is open on the left : $I = ]a, b >$, and that there exists $y \in I$ such that $\lim_{x \to a}(xTy) \geq y$. We know that $x \mapsto xTy$ is increasing, so that $\forall x \in I,\ xTy > y$. This implies that every element of $I$ is of type (1). So, we have: $\forall y \in I, \lim_{x \to a}(xTy) \geq y$.
Using this remark, we can only consider the three following cases

- Case 1 : $I = [a, b >$. Example : $([1, \infty[, \times)$.
- Case 2 : $I = ]a, b >$ and $\forall y \in I,\ \lim_{x \to a} xTy \geq y (> a)$. Example : $(]1, \infty[, \times)$.
- Case 3 : $I = ]a, b >$ and $\forall y \in I,\ \lim_{x \to a} xTy < y$. Example : $(]0, \infty[, \times)$.

We first claim that Cases 1 and 2 can be viewed as equivalent. Assume that we are in Case 2, and let's define $\tilde{T}$ by

$$\begin{aligned}
a\tilde{T}y &= \lim_{x \to a}(xTy) \\
x\tilde{T}a &= \lim_{y \to a}(xTy) \\
a\tilde{T}a &= \lim_{x \to a} \lim_{y \to a}(xTy)
\end{aligned}$$

It is easily seen that the relation $\tilde{T}$ remains continuous and associative. Let's show that it remains stricly increasing. Assume that $s > t$. Because we are in Case 2, we know that $a\tilde{T}y > a$. Thus $sT(a\tilde{T}y) > tT(a\tilde{T}y)$, because $T$ is increasing. By associativity, we get that $(s\tilde{T}a)Ty > (s\tilde{T}a)Ty$. So, $s\tilde{T}a > t\tilde{T}a$.

Finally, the relation $\tilde{T}$ defined on $[a, b>$ has the same properties as $T$ in Case 1. Then, Cases 1 and 2 can be treated simultenaously.
We will split Cases 1 and 2 in two subcases

- Case 1' : $I = [a, b>$ and $aTa > a$.
- Case 2' : $I = [a, b>$ and $aTa = a$.

In Case 1', we have for all $y \in I, aTy > y$. Thus, every element of $I$ is of type (1). In Case 2', we have for all $y \neq a$, $aTy > a$. This implies that in this case, $a$ is of type (2) and all the other elements are of type (1).

Let's now deal with Case 3. We have in this case

$$\lim_{y \to a} \big( \lim_{x \to a} (xTy) \big) \leq \lim_{y \to a} y = a.$$

This implies immediately that

$$\lim_{y \to a} \lim_{x \to a} xTy = a.$$

We claim that for all $y \in I$, we have

$$\lim_{x \to a} (xTy) = a.$$

Assume that this limit was $a' > a$. We obtain, by our assumption,

$$\begin{aligned} a' > \lim_{x \to a} (xTa') &= \lim_{x \to a} xT(\lim_{s \to a} sTy) \\ &= \lim_{x \to a} \lim_{s \to a} xT(sTy) \\ &= \big( \lim_{x \to a} \lim_{s \to a} (xTs) \big) Ty \\ &= (\lim_{t \to a} t) Ty \\ &= a' \end{aligned}$$

This would be a contradiction.
Take $c \in I$ an element of type (1) : $cTc > c$. We know that $T$ is continuous, $cTc > c$ and $\lim_{x \to a} (xTc) = a < c$. This implies that there exists $e \in ]a, c[$ such that $eTc = c$.
Moreover, we have $eTc = c > e$ and $\lim_{x \to a} (xTc) = a < e$. This implies that there exists $c^{-1} \in ]a, e[$ such that $c^{-1}Tc = e$.

We finally obtain three cases

- Case 1' : $I = [a, b>$, all elements are of type (1).

- Case 2' : $I = [a, b >$, all elements but $a$ are of type (1), $a$ is the identity element.
- Case 3 : $I = ]a, b >$, there exists an identity element $e \in ]a, b[$, all elements above $e$ are of type (1), all elements below $e$ are of type (3). Moreover, each element of type (1) has an inverse in $I$ of type (3) and vice versa.

(3) **Construction of the semigroup homomorphism $f$ in Case 3**

We begin by treating the Case 3, because the construction of $f$ is quite easy when $(I, T)$ is a group.

We define on $I$ for any $m \in \mathbb{N}^*$ the function $\psi_m$ defined by : $\psi_m(u) = uTu \dots Tu$ where the element $u$ appears $m$ times in the formula. We can see that $\psi_{m_1} T \psi_{m_2} = \psi_{m_1+m_2}$.

We know that $\lim\limits_{u \to a} \psi_m(u) = a$ and $\lim\limits_{u \to b} \psi_m(u) = b$. Thus, for any $m \geq 1$, the function $\psi_m^{-1} : ]a, b[ \to ]a, b[$ is well-defined. It can be also seen that this function preserves the types.

Take an arbitrary element $c$ of type (1) in $I : c > e$. For $m, n \geq 1$, we define

$$f(\frac{m}{n}) = \psi_m(\psi_n^{-1}(c)).$$

We obtain the following properties on $f : \forall m, m_1, m_2, n \geq 1$,

$$f(\frac{m_1}{n}) T f(\frac{m_2}{n}) = \psi_{m_1}(\psi_n^{-1}(c)) T \psi_{m_2}(\psi_n^{-1}(c)) = \psi_{m_1+m_2}(\psi_n^{-1}(c)) = f(\frac{m_1 + m_2}{n})$$
$$f(\frac{m+1}{n}) = f(\frac{m}{n}) T f(\frac{1}{n}) > f(\frac{m}{n}) T e = f(\frac{m}{n})$$

Thus, the function $f$ defined on the positive rationals is an increasing homomorphism. What's going on in 0? Assume that $\lim\limits_{n \to \infty} f(\frac{1}{n}) = d > e$. Then, we have

$$d = \lim_{n \to \infty} f(\frac{1}{n}) = \lim_{n \to \infty} f(\frac{1}{2n}) T \lim_{n \to \infty} f(\frac{1}{2n}) = dTd > d.$$

This leads to a contradiction. So, $\lim\limits_{n \to \infty} f(\frac{1}{n}) = e$. We define $f(0) = e$.

What's going on in $\infty$? Assume that $\lim\limits_{n \to \infty} f(n) = b' \in I$. Then, we have $b' < cTb'$, and we can conclude easily that $b' < b'$, which is a contradiction. Thus, $\lim\limits_{n \to \infty} f(n) = b$ and $I$ is open on the right.

Take $x \in \mathbb{R}^*{}_+$, we define $f(x)$ by

$$f(x) := \sup\{f(r) | r \in \mathbb{Q} \cap ]-\infty, x[\} = \inf\{f(R) | R \in \mathbb{Q} \cap ]x, \infty[\}.$$

The last equality comes from the following reasoning: denote by $u$ the superior limit and by $v$ the inferior limit above. Assume $u < v$, then $uTu < uTv < vTv$, there must exists $\varepsilon > 0$ such that

$$uTu + \varepsilon < uTv < vTv - \varepsilon$$

Because $T$ is continuous, there exists $\delta > 0$, such that for all $u' \in B(u, \delta), v' \in B(v, \delta)$,

$$uTu + \varepsilon < u'Tv' < vTv - \varepsilon$$

There exists some rationals $r < x < R$ such that $u' = f(r) \in B(u, \delta)$, $v' = f(R) \in B(v, \delta)$ and $\frac{r+R}{2} > x$. We have then

$$\psi_2[f(\frac{r+R}{2})] = f(r+R) = f(r)Tf(R) = u'Tv' < uTv + \varepsilon < vTv = \psi_2(v)$$

This contradicts the fact that : $\psi_2$ is increasing and that $v$ is the inferior limit.
For any $x \in \mathbb{R}_-$, we define : $f(x) = f(-x)^{-1}$.
Finally, we claim that the function $f$ is continuous, increasing, such that for all $x, y \in \mathbb{R}$, $f(x)Tf(y) = f(x+y)$ and $f(\mathbb{R}) = I$.

The construction of $f$ in Case 2' is analogous.

(4) **Construction of the semigroup homomorphism $f$ in Case 1' (very technical)**
We use the functions $\{\psi_m\}$ previously introduced. Notice that in this case, because every element is of type (1), the sequence $(\psi_m(x))$ is increasing for all $x \in I$.
For a given $m$, the function $\psi_m$ is continuous and $\lim_{x \to b} \psi_m(x) = b$. Thus the equation in $x$ : $\psi_m(x) = y$ has a unique solution for every $y \in [\psi_m(a), b[$. The inverse function $\psi_m^{-1}$ is well-defined on this interval.
We define $f(1) = a$, and for $m \geq n \geq 1$,

$$f(\frac{m}{n}) = \psi_n^{-1}(\psi_m(a)).$$

We show that $f$ is an homomorphism.
Take $p, q \geq n$. We have

$$f(\frac{pq}{n}) = \psi_n^{-1}[\psi_p(\psi_q(a))].$$

Because $\psi_q(a) > \psi_n(a)$, we can commute $\psi_n^{-1}$ and $\psi_p$. Thus,

$$f(\frac{pq}{n}) = \psi_p[\psi_n^{-1}(\psi_q(a))].$$

Take $p \geq n$, we have

$$
\begin{aligned}
f(p)Tf\big(\frac{p}{n}\big) &= \psi_p(a)T\psi_n^{-1}(\psi_p(a)) \\
&= \psi_n[\psi_n^{-1}(\psi_p(a))]T\psi_n^{-1}(\psi_p(a)) \\
&= \psi_{n+1}[\psi_n^{-1}(\psi_p(a))] \\
&= \psi_n^{-1}[\psi_{n+1}(\psi_p(a))] \\
&= f(\frac{p+1}{n}) \\
&= \psi_p[\psi_n^{-1}(\psi_{n+1}(a))]
\end{aligned}
$$

For the same reason, $f\big(\frac{p}{n}\big)Tf(p) = f\big(\frac{p+1}{n}\big)$, we obtained

$$
f(p)Tf\big(\frac{p}{n}\big) = f\big(\frac{p}{n}\big)Tf(p).
$$

We have for $m_1, m_2 \geq n$,

$$
\begin{aligned}
f(m_1)Tf\big(\frac{m_1}{n}\big)Tf\big(\frac{m_2}{n}\big)Tf(m_2) &= \psi_{m_1}[\psi_n^{-1}(\psi_{n+1}(a))]T\psi_{m_2}[\psi_n^{-1}(\psi_{n+1}(a))] \\
&= \psi_{m_1+m_2}[\psi_n^{-1}(\psi_{n+1}(a))] \\
&= f\big(\frac{m_1+m_2}{n}\big)Tf(m_1+m_2) \\
&= f\big(\frac{m_1+m_2}{n}\big)Tf(m_1)Tf(m_2) \\
&= f(m_1)Tf(m_2)Tf\big(\frac{m_1+m_2}{n}\big)
\end{aligned}
$$

By reducibility of $T$, this leads to

$$
\begin{aligned}
f\big(\frac{m_1+m_2}{n}\big)Tf(m_1) &= f(m_1)Tf\big(\frac{m_1}{n}\big)Tf\big(\frac{m_2}{n}\big) \qquad (*) \\
&= f\big(\frac{m_1}{n}\big)Tf(m_1)Tf\big(\frac{m_2}{n}\big) \\
f(m_2)Tf\big(\frac{m_1+m_2}{n}\big) &= f\big(\frac{m_1}{n}\big)Tf(m_2)Tf\big(\frac{m_2}{n}\big)
\end{aligned}
$$

Taking $p = m_1$, $m_2 = rn$, and then $p = m_2$, $m_1 = rn$, we get for every $p, r \in \mathbb{N}$,

$$
\begin{aligned}
f\big(\frac{p}{n}+r\big)Tf(p) &= f\big(\frac{p}{n}\big)Tf(p)Tf(r) \\
&= f\big(\frac{p}{n}\big)Tf(r)Tf(p) \\
f(p)Tf\big(r+\frac{p}{n}\big) &= f(r)Tf(p)Tf\big(\frac{p}{n}\big) \\
&= f(p)Tf(r)Tf\big(\frac{p}{n}\big)
\end{aligned}
$$

We have

$$
f\big(\frac{p}{n}+r\big) = f\big(\frac{p}{n}\big)Tf(r) = f(r)Tf\big(\frac{p}{n}\big)
$$

Replacing this in (*), we finally obtain :

$$
f\big(\frac{m_1+m_2}{n}\big)Tf(m_1) = f\big(\frac{m_1}{n}\big)Tf\big(\frac{m_2}{n}\big)Tf(m_1).
$$

The extension of $f$ on the set $]1,\infty[$ can be done as before, $f$ keeps the desired properties.

# References


[1] Arrow, K.J., Sen, A.K., Suzumura, K. (Eds.). (2002). *Handbook of Social Choice and Welfare,* Vol 1. North-Holland.

[2] Gorman, W.M. The strucure of utility functions. *Review of Economic Studies*, 32 (1968), pp. 369-390.

[3] Debreu, G. (1959). *Theory of value,* Wiley, New York.

[4] Aczel, J. (1966). *Lectures on functional equations and their applications,* Academic Press, New York and London.